0.1 Additive Rules

- **Theorem 2.10**:
  If A and B are any two events, then
  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Corollary 1**:
  If A and B are mutually exclusive, then
  $$P(A \cup B) = P(A) + P(B)$$

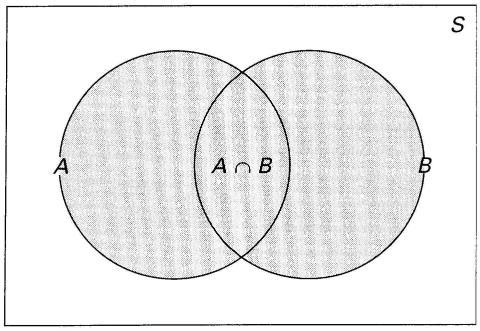

Figure 1: Additive rule of probability.

- **Corollary 2**:
  If $A_1, A_2, \ldots A_n$, are mutually exclusive, then
  $$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

- **Corollary 3**:
  If $A_1, A_2, \ldots A_n$, , is a partition of a sample space S, then
  $$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$
  $$= P(S) = 1$$

- **Theorem 2.11**: (an extension of Theorem 2.10)

For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$$
$$-P(B \cap C) + P(A \cap B \cap C)$$

- **Theorem 2.12**:

If A and A' are complementary events, then

$$P(A) + P(A') = 1$$

Proof : Since $A \cup A' = S$ and $A \cap A' = \varnothing$, then

$$1 = P(S) = P(A \cup A') = P(A) + P(A')$$

- **Example 2.32**: The probability that the production procedure meets specification (2000 ± 10 *mm*) is known to be 0.99. Small cable is just as likely to be defective as large cable.
  - What is the probability that a cable selected randomly is too large?
    Let M be the event that a cable meets spec. Let S and L be the events that the cable is too small and too large, respectively. Then
    $$P(M) = 0.99 \text{ and } P(S) = P(L) = (1 - 0.99)/2 = 0.0005$$
  - What is the probability that a cable selected randomly is larger than 1990 mm?
    $$P(X \geq 1990) = 1 - P(S) = 0.995$$
    where X is the length of a randomly selected cable.

0.2 Conditional Probability

- **Conditional probability:** $P(B|A)$
  - Sometimes the occurrence of an event is influenced or related with some other event.
  - Hence we must take this *restriction* or *the availability of certain limited information* into consideration about the outcome of the experiment.

- The probability of an event B occurring when it is known that some event A has occurred.
- "The probability that B occurs given that A occurs" or "The probability of B,given A"
- The notion of conditional probability provides the capability of re-evaluating the idea of probability of an event in light of additional information.

- **Example**: $S = 1, 2, 3, 4, 5, 6,\ \ A = 4, 5, 6,\ \ B = 1, 3, 5, \Longrightarrow P(B|A)$?

- **Definition 2.9**:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

provided $P(A) > 0$

- **Example**: Our sample space S is the population of adults in a small town. They can be categorized according to gender and employment status (see Table 1).

Table 1: Categorized adult population in a small town.

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

- One individual is to be selected at random for a publicity tour.
- The concerned events
  - M: a man is chosen
  - E: the one chosen is employed

$$P(M|E) = \frac{460}{600} = \frac{23}{30}$$

$$P(M|E) = \frac{n(E \cap M)/n(S)}{n(E)/n(S)} = \frac{P(E \cap M)}{P(E)} = \frac{\frac{460}{900}}{\frac{600}{900}} = \frac{23}{30}$$

- **Example 2.33**: The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$;

- the probability that arrives on time is $P(A) = 0.82$;
- the probability that it departs and arrives on time is $P(D \cap A) = 0.78$.
- Find the probability that a plane
  - arrives on time given that it departed on time, and

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

  - departed on time given that it has arrived on time.

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

- **Definition 2.10**:

> Two events A and B are said to be **independent** if and only if
>
> $$P(B|A) = P(B) \text{ or } P(A|B) = P(A).$$
>
> Otherwise, A and B are **dependent**.

- If knowing that event B occurred doesn't change the probability that A will occur, then B must carry no information about A.
- The condition $P(B|A) = P(B)$ implies that $P(A|B) = P(A)$, and conversely.
- **Example**: Two cards are drawn in succession, with replacement
  - Event A: the first card is an ace
  - Event B: the second card is a spade

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/52}{4/52} = \frac{13}{52} = \frac{1}{4} \text{ and } P(B) = \frac{13}{52} = \frac{1}{4}$$

  - Since $P(B|A) = P(B)$, these two events are independent.

0.3 Multiplicative Rules

- Multiplying the formula of Definition 2.9 by $P(A)$, we obtain the **multiplicative rule**, which enables us to calculate the probability that two events will both occur.

- **Theorem 2.13**:

> If in an experiment the events A and B can both occur, then
> $$P(A \cap B) = P(A) * P(B|A)$$
> provided $P(A) > 0$

- We can also write
$$P(A \cap B) = P(B \cap A) = P(B) * P(A|B)$$

- **Example 2.35**: Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession *without replacing* the first.

- What is the probability that both fuses are defective?
  - Event A: the first fuse is defective
  - Event B: the second fuse is defective. Hence,
$$P(A \cap B) = P(A) * P(B|A) = \frac{1}{4} * \frac{4}{19} = \frac{1}{19}$$

- **Example 2.36**: One bag contains 4 white balls and 3 black balls. A second bag contains 3 white balls and 5 black balls.

- One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

- Solution: Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1.
$$\begin{aligned} p[(B_1 \cap B_2) \cup (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \frac{3}{7} * \frac{6}{9} + \frac{4}{7} * \frac{5}{9} = \frac{38}{63} \end{aligned}$$

- **Theorem 2.14**:

> Two events A and B are (statistically or probabilistically) independent if and only if
> $$P(A \cap B) = P(A)P(B)$$
> . Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

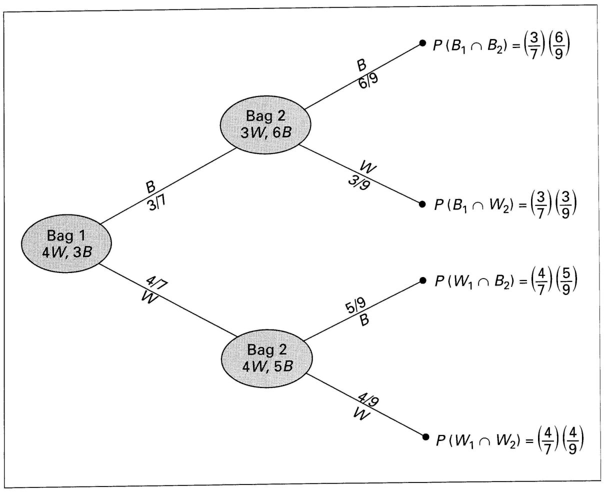

Figure 2: Tree diagram for Example 2.36.

- **Example 2.37**: A small town has one fire engine and one ambulance available for emergencies.
  - The probability that the fire engine is available when needed is 0.98,
  - The probability that the ambulance is available when called is 0.92
  - In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available.
- **Solution**: Let A and B represent the respective evens that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B) = 0.98 * 0.92 = 0.9016.$$

- **Example 2.38**: Find the probability that
  - the entire system works
  - the component C does not work, given that the entire system works
- Solution:

$$P(A \cap B \cap (C \cup D)) = P(A) * P(B) * P(C \cup D)$$

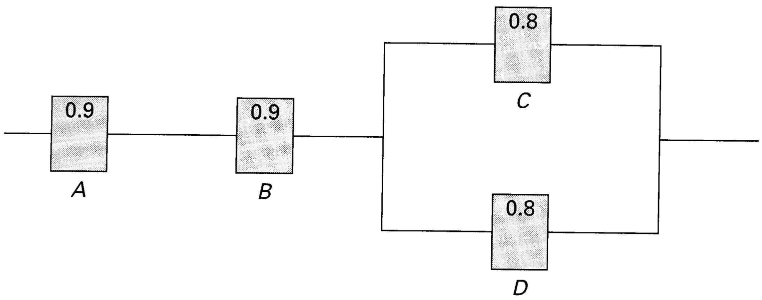

Figure 3: An electrical system for Example 2.38.

$$= P(A) * P(B) * (1 - P(C' \cap D')) = P(A) * P(B) * (1 - P(C') * P(D'))$$
$$= 0.9 * 0.9 * (1 - (1 - 0.8) * (1 - 0.8)) = 0.7776$$

- $$P = \frac{P(\textit{the system works but } C \textit{ does not work})}{P(\textit{the system works})}$$
$$= \frac{P(A \cap B \cap C' \cap D)}{P(A \cap B \cap (C \cup D))} = \frac{0.9 * 0.9 * (1 - 0.8) * 0.8}{0.7776} = 0.1667$$

- **Independence** is often easy to grasp intuitively.

- For example, if the occurrence of two events is governed by distinct and non-interacting physical processes, such events will turn out to be independent.

- On the other hand, independence is not easily visualized in terms of the sample space.

- A common fallacy (wrong idea) is that two events are independent if they are disjoint, but in fact *the opposite is true*:

Two disjoint events A and B with $P(A) > 0$ and $P(B) > 0$ are never independent, since their intersection $A \cap B$ is empty and has probability 0.

- We note that

  (i) independent events are never mutually exclusive,

  (ii) two mutually exclusive events are always dependent.

- **Theorem 2.15**:

  If the events $A_1, A_2, A_3, \ldots, A_k$ can occur, then

  $$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

  $$\ldots P(A_k|A_1 \cap A_2 \cap \ldots \cap A_k)$$

  If the events $A_1, A_2, A_3, \ldots, A_k$ are independent, then

  $$P(A_k|A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)P(A_2)\ldots P(A_k) = \prod_{n=1}^{k} P(A_n)$$

- **Example 2.39**: Three cards are drawn in succession without replacement. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where
  - A_1: the first card is red ace
  - A_2: the second card is a 10 or jack
  - A_3: the third card is greater than 3 but less than 7

- **Solution**:

  $$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

  $$= \frac{2}{52} * \frac{8}{51} * \frac{12}{50} = \frac{8}{5525}$$

- **Independence of Several Events**:

  The events $A_1, A_2, A_3, \ldots, A_n$ are **independent** if

  $$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$$

  for any subset S of $\{1, 2, \ldots, n\}$.

- Independence means that the occurrence or non-occurrence of any number of the events from that collection carries no information on the remaining events or their complements.

- **Example: Independence of three events**: If A_1, A_2 and A_3 are independent,

  $$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

  $$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

- **Example**: Consider two independent fair coin tosses, and the following events:
  - H_1= 1^{st} toss is a head,
  - H_2= 2^{nd} toss is a head,
  - D = the two tosses have different results.
- Pairwise independence does not imply independence.
  - H_1 and H_2 are independent, by definition.
  - $P(D|H_1) = P(D)$ and $P(D|H_2) = P(D)$
  - $P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1)P(H_2)P(D)$
- **Example**: Consider two independent rolls of a fair die, and the following events:
  - $A = 1^{st}$ roll is 1, 2, or 3, $B = 2^{nd}$ roll is 3, 4, or 5, $C =$ the sum of the two rolls is 9.
- $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ is not enough for independence.
  - $P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} * \frac{1}{2} = P(A)P(B)$
  - $P(A \cap C) = \frac{1}{36} \neq \frac{1}{2} * \frac{4}{36} = P(A)P(C)$
  - $P(B \cap C) = \frac{3}{6} \neq \frac{1}{2} * \frac{4}{36} = P(B)P(C)$
  - $P(A \cap B \cap C) = \frac{1}{36} \neq \frac{1}{2} * \frac{1}{2} * \frac{4}{36} = P(A)P(B)P(C)$

0.4 Bayes'Rules

- Our sample space S is the population of adults in a small town. They can be categorized according to employment status.
- One individual is to be selected at random for a publicity tour.
  - The concerned event E: the one chosen is employed
  - Give the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club.
  - Find the probability of the event A that individual selected is a member of the Rotary Club.

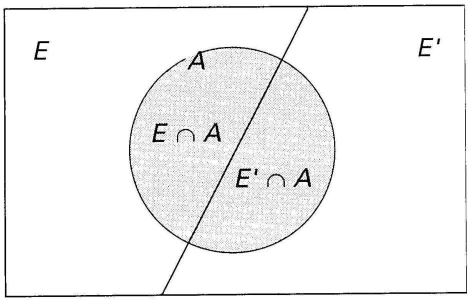

Figure 4: Venn diagram for the events A, E, and E'.

- Event A is the union of the two mutually exclusive events $E \cap A$ and $E' \cap A$. Hence,

$$\begin{aligned} \bullet A &= (E \cap A) \cup (E' \cap A) \\ \bullet P(A) &= P[(E \cap A) \cup (E' \cap A)] \\ &= P(E \cap A) + P(E' \cap A) \\ &= P(E)P(A|E) + P(E')P(A|E') \end{aligned}$$

$$\begin{aligned} \bullet P(E) &= \frac{600}{900} = \frac{2}{3},\ P(A|E) = \frac{36}{600} = \frac{3}{50} \\ \bullet P(E') &= \frac{1}{3}, P(A|E) = \frac{12}{300} = \frac{1}{25} \\ \bullet P(A) &= \frac{2}{3} * \frac{3}{50} + \frac{1}{3} * \frac{1}{25} = \frac{4}{75} \end{aligned}$$

- **Theorem 2.16**: (**Theorem of total probability** or **rule of elimination**)

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

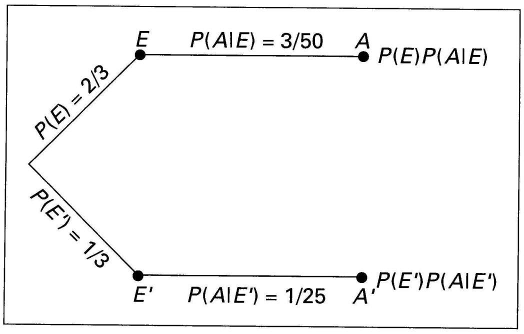

Figure 5: Tree diagram for the data.

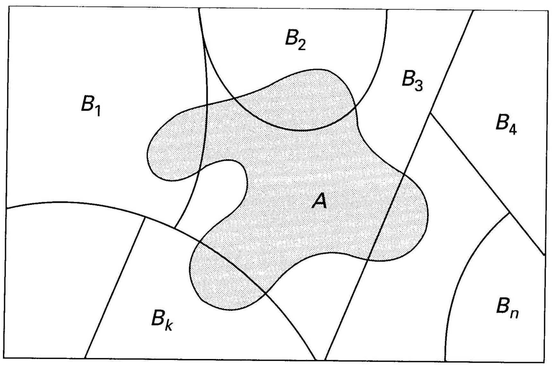

Figure 6: Partitioning the sample space S.

- **Example 2.41**: In a certain assembly plant, three machines, B_1, B_2 and B_3 make 30%, 45% and 25%, respectively, of the products.
- It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.
- Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$

$$= 03 * 0.02 + 0.45 * 0.03 + 0.25 * 0.02 = 0.0245$$

- Solution:
- Event A: the product is defective.
- Event B: the product is made by machine B_i

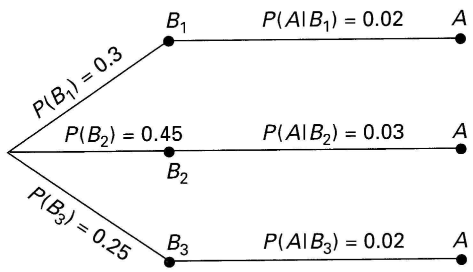

Figure 7: Tree diagram for Example 2.41.

- **Theorem 2.17**: (Bayes'Rule)

> If the events $B_1, B_2, \ldots B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then
>
> $$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

- It can be proved by the definition of conditional probability,

$$P(B_r|A) = P(B_r \cap A)/P(A)$$

and then using Theorem 2.16 in the denominator.

- Useful in problems where $P(B_i|A)$ are not known but $P(A|B_i)$ and $P(B_i)$ are known.

- Some terminology:
  - $P(B_i)$: priors
  - $P(A|B_i)$: likelihoods
  - $P(B_i|A)$: posteriors

- **Example 2.42**: With reference to Example 2.41, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3

- Using Bayes'rule,

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{10}{49}$$

- **Example 2.43**: A manufacturing firm employs three analytical plans for the design and development of a particular product.
- For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products respectively.
- The "defect rate" is different for the three procedures as follows:

$$P(D|P_1) = 0.01,\ P(D|P_2) = 0.03,\ P(D|P_3) = 0.5$$

  where $P(D|P_j)$ is the probability of a defective product, given plan j.

- If a random product was observed and found to be defective, which plan was most likely used and thus responsible?
- **Solution**: $P(P_1) = 0.3, (P_12) = 0.2, (P_1) = 0.5$

$$P(P_i|D) = \frac{P(P_i)P(D|P_i)}{\sum_{i=1}^{3} P(P_i)P(D|P_i)} = \frac{(0.30)(0.01)}{(0.3)(0.01) + (0.20)(0.03) + (0.50)(0.02)} = \frac{0.003}{0.019}$$

$$P(P_1|D) = 0.158,\ P(P_2|D) = 0.316,\ P(P_3|D) = 0.526.$$