Chapter 10 - Rotation

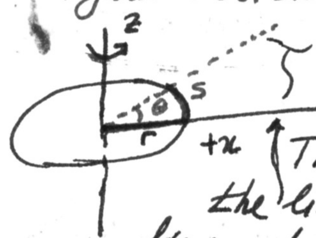
Motion of translation → along a straight line

Motion of rotation $\xrightarrow{\text{a rigid body}}$ turns around an axis (about COM!) SLN

- → rotational acceleration (constant or not!)
- → torque (instead of force)
- → inertia (instead of mass)

- Rotational Variables A rigid body about a fixed axis SLN Fig. 10-2

1) Angular Position $\theta(t)$ time dependence



reference line, perpendicular to rotation axis

The angular position of the line is the angle of the line relative to a fixed direction.

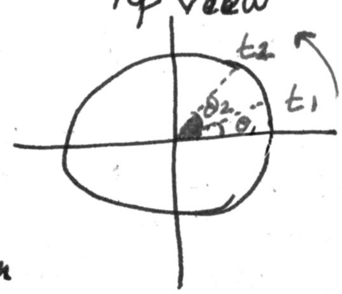
zero angular position

$\boxed{\theta = \frac{s}{r}}$ circular arc length / radius of circle } • radian measure } angle

$360° = 2\pi \text{ rad} \equiv 1 \text{ revolution}$

2) Angular Displacement $\Delta\theta$

Top view



$\boxed{\Delta\theta = \theta_2 - \theta_1}$

ccw → (+)
cw → (−)

3) Angular Velocity ω (rad/s or rev/s)

average: $\boxed{\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}}$ instantaneous: $\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$

- ccw → (+)
- cw → (−)

- magnitude is called angular speed, ω

4) Angular Acceleration α (rad/s² or rev/s²)

$\boxed{\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{\Delta t}}$ $\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}}$

→ $\theta = \int \omega\, dt$ $\omega = \int \alpha\, dt$

Example Angular velocity derived from angular position

Disk is rotating as $\theta(t) = -1.00 - 0.600t + 0.250t^2$ SLN Fig. 10-5a

i) Angular position of reference line at $t = -2.0\text{s}, 0\text{s}, 4\text{s}$, $\theta = 0$ points

$t=-2 \rightarrow \theta(t=-2) = -1 - 0.6(-2) + 0.25(-2)^2 = 1.2 \text{ rad} \Rightarrow$ 2π rad 360°, 1.2 rad x } $\theta = 69°$ ccw

SLN Fig. 10-5b for the rest } $t=0 \rightarrow \theta = -1.00 \text{ rad} \rightarrow -57°$ cw; $t=4 \rightarrow \theta = 0.60 \text{ rad} \rightarrow 34°$ ccw

} $\theta = 0$ points: reference line is aligned with zero angular position (s2, s4)

ii) $t_{min} = ?$ $\xrightarrow{\text{that makes}}$ $\theta(t)$ minimum.

SLN. Fig. 10-5c → What about angular acceleration!

To have a minimum $\left.\frac{d\theta}{dt}\right|_{t=t_{min}} = 0 \rightarrow -0.6 + 0.5t = 0 \rightarrow t = 1.20\text{s}$ (see Fig.10-5b)

$\theta(t=1.20\text{s}) = -1.36 \text{rad} \sim -77.9°$ maximum cw rotation!

iii) $t=0 \rightarrow \omega(0) = -0.6$ rad/s
$t=1 \rightarrow \omega(1) = -0.1$ "
$t=2 \rightarrow \omega(2) = 0.4$ "
} $\frac{d\theta}{dt} = \omega = -0.6 + 0.5t$

Example Angular velocity derived from angular acceleration

$\alpha = 5t^3 - 4t$

$t=0 \begin{cases} \omega = 5 \text{ rad/s} \\ \theta = 2 \text{ rad} \end{cases}$

i) $\omega(t) = ?$ $\int d\omega = \int \alpha\, dt \rightarrow \omega = \int (5t^3 - 4t)dt = \frac{5}{4}t^4 - \frac{4}{2}t^2 + C$.

$\omega(t=0) = 5 = \frac{5}{4}0^4 - \frac{4}{2}0^2 + C \Rightarrow \omega(t) = \frac{5}{4}t^4 - 2t^2 + 5$

ii) $\theta(t) = ?$ $\int d\theta = \int \omega\, dt \rightarrow \theta = \int (\frac{5}{4}t^4 - 2t^2 + 5)dt = \frac{1}{4}t^5 - \frac{2}{3}t^2 + 5t + C$

$\theta(t=0) = 2 \rightarrow \theta(t) = \frac{t^5}{4} - \frac{2}{3}t^2 + t + 2$

- Are Angular Quantities vectors?

Angular Displacement, $\Delta\theta \rightarrow$ Can not be treated as vectors. Does not obey to vector arithmetics.

Angular Velocity, ω } Can be treated as vectors { SLN Fig. 10-6 Fig. 10-7 Directions of vector and motion are different } ω and α can be represented by $\pm$ sign. ccw (+) cw (-)

Angular Acceleration, α

SLN Rotation with Constant Angular Acceleration Table 10-1

Example Constant angular acceleration, grindstone

$\alpha = 0.35 \text{ rad/s}^2$

$\omega_0 = -4.6 \text{ rad/s}$

$\theta_0 = 0$ (reference line)

SLN Fig. 10-8

i) $t = ?$ at $\theta = 5$ rev ($\theta - \theta_0$) $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ $5 \times 2\pi \text{ rad} = -4.6 \text{ rad/s}\, t + 0.35 \text{ rad/s}^2\, t^2$

$\Rightarrow t = 32$ s

ii) $\alpha \rightarrow$ positive, $\omega_0 \rightarrow$ negative } initially slows down, momentarily stops, rotates again cw, ccw since α (+) θ (+)

iii) $t = ?$ at $\omega = 0$ $\cancel{\omega} = \omega_0 + \alpha t \rightarrow +4.6 \text{ rad/s} = 0.35 \text{ rad/s}^2\, t$

$\Rightarrow t = 13$ s

Example Constant angular acceleration, riding a Rotor

$\omega_0 = 3.4 \text{ rad/s}$

$\omega = 2.0 \text{ rad/s}$

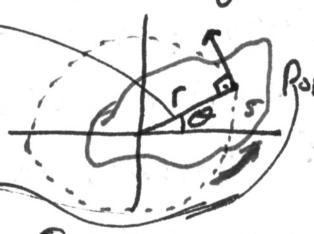
$\theta - \theta_0 = 20.0 \text{ rev} (\times 2\pi \text{ rad/rev})$

constant angular acceleration

i) $\alpha = ?$ $\omega = \omega_0 + \alpha t$, $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ } $\theta - \theta_0 = \omega_0\left(\frac{\omega - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2 \Rightarrow \alpha = -0.0301 \text{ rad}$ slowing down

ii) $t = \frac{\omega - \omega_0}{\alpha} = \frac{2.0 \text{ rad/s} - 3.4 \text{ rad/s}}{-0.0301 \text{ rad/s}^2} = 46.5$ s

Fig. 10-9a



Relating the Linear and Angular Variables

s, v, a } can be related to angular counterparts { θ, ω, α by r: the perpendicular distance of the point from the rotation axis

Point P makes a rotation. velocity v, distance s

Object makes a rotation about a fixed axis. ω } $s = \theta r$, $v = \omega r$ (linear speed, angular speed)

$\Rightarrow$ linear speed v depend on the "point"'s location

angular speed ω is same at every "point"

$T = \frac{2\pi r}{v} \rightarrow \boxed{T = \frac{2\pi}{\omega}}$ $2\pi r \leftrightarrow \theta r$: distance travelled

$s = \theta r$

$\frac{ds}{dt} = \frac{d\theta}{dt} r \rightarrow v = \omega r$

$\frac{dv}{dt} = \frac{d\omega}{dt} r \rightarrow a = \alpha r$

SLN Fig. 10-9b

$a \Rightarrow a_t$: tangential component

Remember

$a_r = \frac{v^2}{r} = \omega^2 r$

radially inward (for changes in the direction of linear velocity)

a_t is present when $\alpha \neq 0$

a_r is present when $\omega \neq 0$

Kinetic Energy of Rotation

$KE = \frac{1}{2} \cancel{M v^2}$

since $v_{com} = 0$

Instead

Suppose that the body is composed of many particles. Then $K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v^2 + \dots + \frac{1}{2} m_n v_n^2$

$$\boxed{K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2}$$

$v = \omega r \Rightarrow K = \frac{1}{2} \sum_{i=1}^{n} m_i \omega^2 r_i^2$

ω same for all particles

$$\Rightarrow \boxed{K = \frac{1}{2} \Big(\underbrace{\sum_{i=1}^{n} m_i r_i^2}_{I\text{ : rotational inertia}}\Big) \omega^2 = \frac{1}{2} I \omega^2}$$

Kinetic Energy of a rigid body in pure rotation

- Tells us how the mass of rotating body is distributed about its axis of rotation.
- It is specified with respect to rotation axis. S&N Fig. 10-11
- $kg\,m^2$
- $\underbrace{\text{Smaller } I}$ means easier rotation

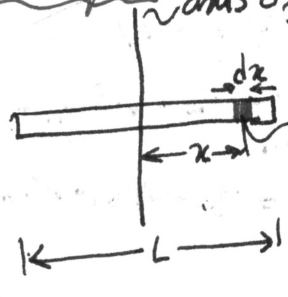
  Mass distribution is close to rotation axis.

Kinetic Energy of the body in pure translation $\rightarrow K = \frac{1}{2} M v_{com}^2$

Calculating the Rotational Inertia

A rigid body consists
- of a few particles $\rightarrow I = \sum m_i r_i^2$ — perpendicular distance from rotation axis
- of a great many adjacent particles $\rightarrow I = \int r^2 dm$: continuous body

Example.



$\frac{M}{L} = \lambda = \frac{dm}{dx}$

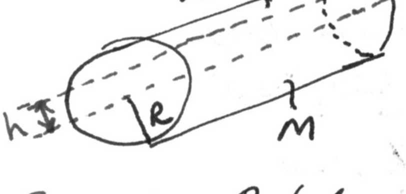
$dm = \lambda dx = \frac{M}{L} dx$

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L} \left(\frac{x^3}{3}\right)\Bigg|_{-L/2}^{L/2}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

for thin rod about axis through center perpendicular to length (see Table 10-2e)

- Parallel Axis Theorem



If we know I about an axis (com axis), then I_{com} we can calculate I about an another axis parallel to first one.

$$I_{axis\,1} = \frac{1}{2} M R^2 = I_{com} \Longrightarrow \boxed{I = \underbrace{\frac{1}{2} M R^2}_{I_{com}} + M h^2}$$

Parallel axis theorem

Example Rotational Inertia of a two particle system. Fig. 10-13a

i) Rotational axis $\rightarrow$ com axis $\quad I = \sum_{i=1}^{2} m_i r_i^2 = m_1 \left(\frac{L}{2}\right)^2 + m_2 \left(\frac{L}{2}\right)^2 = \frac{M L^2}{4} = \boxed{\frac{mL^2}{2}}$ ($M = m_1 + m_2$, if $m_1 = m_2$)

ii) Rotational axis $\rightarrow$ at left end $\quad I = I_{com} + M h^2$

$$= \underbrace{\frac{mL^2}{2} + 2m\left(\frac{L}{2}\right)^2}_{\text{by parallel axis theorem}} \boxed{= mL^2}$$

or $I = \sum_{i=1}^{2} m_i r_i^2$

$= m_1 (0)^2 + m_2 L^2 \boxed{= mL^2}$

Torgue, τ: (To twist)

Does not cause rotation

Resolve applied force ($\vec{F}$) for rotation into two components — $\vec{F_r}$: radial component; $\vec{F_t}$: tangential component

Does cause rotation: $|\vec{F}|\sin\phi = \vec{F_t}$

SLN Fig. 10-16 $\quad \tau = rF_t = rF\sin\phi \rightarrow$ Fig. 10-16b

SI Unit: N.m $\quad \tau = (r\sin\phi)F = r_\perp F \rightarrow$ Fig. 10-16c

(Be aware that torque is not work! 1J = 1N.m)

- Rotation around an axis → in 1D ⇒ Sign of torque: (+) ccw; (−) cw
- When several forces acting → several torques ⇒ net torque (τ_{net}) is obtained by superposition principle.

Newton's 2nd law for Rotation SLN Fig. 10-17

Net torque causes an angular acceleration, α. $\boxed{\tau_{net} = I\alpha}$ Newton's 2nd law of rotation (radian measure)

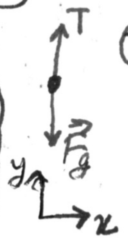
Proof: F_t creates a_t $\}$ $F_t = ma_t$ $\}$ $F_t r = ma_t r$ $\}$ $\tau = m(\alpha r)r$ $\}$ $\tau = (mr^2)\alpha$ $\}$ $\tau = I\alpha$

Example. Newton's 2nd Law in Rotational Motion

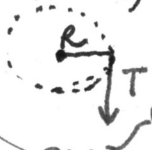
SLN Fig. 10-18

$M = 2.5$ kg
$R = 0.2$ m
$m = 1.2$ kg
$a = ?$, $\alpha = ?$
$T = ?$

i) $a = ?$ Acceleration of falling block



$T^{?} - mg = ma$ ①



$\tau = I\alpha$

cw: $-RT = I\alpha$

$-RT = \frac{1}{2}MR^2\left(\frac{a}{r}\right)$

$\Rightarrow T = -\frac{1}{2}Ma$ ②

Combining ① & ②

$-\frac{1}{2}Ma - mg = ma$

$a(m + \frac{1}{2}M) = -mg$

$a = -\frac{2m}{2m + M}g = \boxed{-4.8\ m/s^2}$

ii) $\alpha = ?$ $\quad \alpha = \frac{a}{r} = \frac{-4.8\ m/s^2}{0.20} = \boxed{-24\ rad/s^2}$

iii) $T = -\frac{1}{2}Ma = -\frac{1}{2}(2.5\,kg)(-4.8\,m/s^2)$

$\boxed{T = 6.0N}$

Work and Rotational Kinetic Energy

Translational	Rotational
F on a rigid body (m) → acceleration (motion) → does work → KE can change	τ on a rigid body → rotational acceleration (motion) → does work → KE can change
$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$	$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$
$W = \int_{x_i}^{x_f} F\,dx$ $\;\}\;$ $P = \frac{dW}{dt} = Fv$	$W = \int_{\theta_i}^{\theta_f} \tau\,d\theta$ $\;\}\;$ $P = \frac{dW}{dt} = \tau\omega$

KE–Work Theorem $\qquad$ SLN Table 10-3

Example Work, Rotational KE, torque, disk SLN Fig. 10-18

$t = 0 \rightarrow \omega = 0$
$T = 6.0N$
$\alpha = -24\ rad/s^2$
KE = ? at $t = 2.5s$
$M = 2.5$ kg
$R = 0.20$ m

$KE = \frac{1}{2}I\omega^2$?

$I = \frac{1}{2}MR^2$ $\quad \omega = \omega_0 + \alpha t$

$\omega = (-24\ rad/s^2)(2.5s)$

$KE = \frac{1}{4}(2.5kg)(0.20m)^2\left[(-24\ rad/s^2)(2.5s)\right]^2$

$\boxed{KE = 90J}$

$W = \tau(\theta_f - \theta_i) = \tau(\omega_0 t + \frac{1}{2}\alpha t^2) = (-TR)(\frac{1}{2}\alpha t^2) = -\frac{1}{2}TR\alpha t^2 = 90J$