1 Probability

1.1 Sample Space

- **Definition**: (Probability theory) The mathematical study of randomness or mechanism of chance.
- In the study of statistics, we are concerned with the presentation and interpretation of **chance outcomes**.
- The outcome will depend on chance and, thus, cannot be predicted with certainty.
- Any recording of information, whether it be numerical or categorical, is *referred to* **observation**.
  - the number of accidents in one month: 2, 0, 1, 2.
  - the category that an inspected item belongs to: D, N, D, N, N.
- **Experiment**: any process that generates (or observe) a set of data.
  - E.g., tossing of a coin, two possible outcomes, heads and tails
  - In a statistical experiment, the data are subject to uncertainty.
- **Definition 2.1**: The set of possible outcomes of a statistical experiment is called the **sample space**,represented by S.
- Each outcome in a sample space is called
  - an **element**,
  - a **member** of the sample space, or
  - a **sample point**.
- If the sample space has a finite number of elements, we may list the members.
- If the sample space has a large or infinite number of elements, we describe it by a **statement** or **rule**.
- **Example 2.1**.
  - Tossing a coin: $S = H, T$
  - Tossing a die:

* $S_1 = \{1, 2, 3, 4, 5, 6\}$
* $S_2 = \{even, odd\}$

- A **tree diagram** can be used to list the elements of the sample space systematically.

- **Example 2.2**. Flip a coin first. If a head occurs, flip it again; otherwise, toss a die.

  - $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$

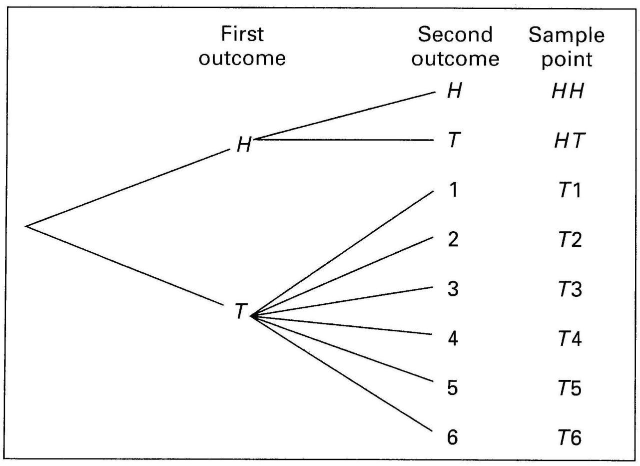

Figure 1: Tree diagram for Example 2.2.

- **Example 2.3**. Three items are selected at random from a process.

  - $S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$

- **The rule method**. The rule method has practical advantages, particularly for the many experiments where a listing becomes a tedious chore.

  - $S = \{x | \ x \ is \ a \ city \ with \ population \ over \ 1 \ million\}$.
  - $S = \{(x, y) | \ x^2 + y^2 \leq 4\}$, the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin.

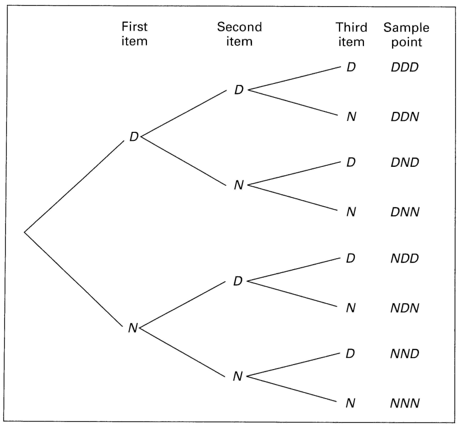

Figure 2: Tree diagram for Example 2.3.

1.2 Events

- **Definition 2.2**: An **event** is a subset of a sample space..
  - **Null set**, denoted $\varnothing$, contains no elements at all.
  - **Example 2.4**: Given the sample space $S = \{t|t \geq 0\}$, where t is the life in years of a certain electronic component.
  - The event A that the component fails before the end of the fifth year is the subset $A = \{t|0 \leq t < 5\}$.
- **Definition 2.3**: The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbolA'.
  - **Example 2.5**: Let R be the event that a red card is selected from an ordinary deck of 52 playing cards.
  - S be the entire deck.
  - R' is the event that the card selected from the deck is not a red but a black card.
- **Definition 2.4**: The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.
- **Example 2.7**: Let P be the event that a person selected at random while dining at a popular cafeteria is a taxpayer.

- Q is the event that the person is over 65 years of age.
- The event $P \cap Q$ is the set of all taxpayers in the cafeteria who are over 65 years of age.

- **Definition 2.5**: Two events A and B are **mutually exclusive**, or **disjoint** if $A \cap B = \varnothing$, i.e., if A and B have no elements in common. Two events can not occur simultaneously.

- **Example 2.9**:
  - Let A be the event that the program belongs to the NBC network.
  - Let B be the event that the program belongs to the CBS network.
  - A and B are mutually exclusive.

- **Definition 2.6**: The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

- **Example 2.12**:
  - If $M = \{x|3 < x < 9\}$ and $N = \{y|5 < y < 12\}$, then
  - $M \cup N = \{x|3 < x < 12\}$

- The relationship between events and the corresponding sample space can be illustrated graphically by **Venn diagram**.

- In a Venn diagram, let the sample space be a rectangle and represent events by circles. In Fig. 3

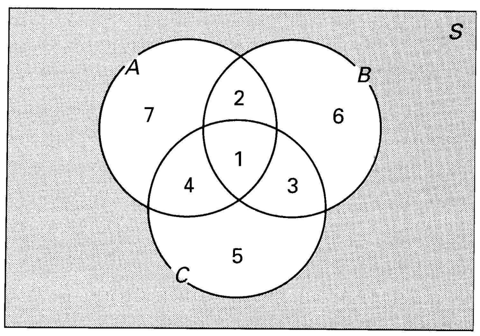

Figure 3: Events represented by various regions.

- $A \cap B$: regions 1 and 2
- $A \cup C$: regions 1, 2, 3, 4, 5, and 7
- $B' \cap A$: regions 4 and 7

1.3 Counting Sample Points

- *Combinatorics* - counting rules in set theory. This provides the idea of the principles of enumeration, counting sample points in the sample space.
- When an experiment is performed, the statistician want to evaluate the chance associated with the occurrence of certain events.
- In many cases we can evaluate the probability by counting the number of points in the sample space.
- **Theorem 2.1 (multiplication rule)**:

  If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

- The multiplication rule is the fundamental principle of counting sample points.
- **Example 2.14**: Home buyers are offered
  - four exterior styling
  - three floor plans
- Since $n_1 = 4, n_2 = 3$ and , a buyer must choose from

$$n_1 n_2 = 12$$

  possible homes

- **Theorem 2.2 (generalized multiplication rule)**:

  If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \ldots n_k$ ways.

- The multiplication rule can be extended to cover any number of operations.

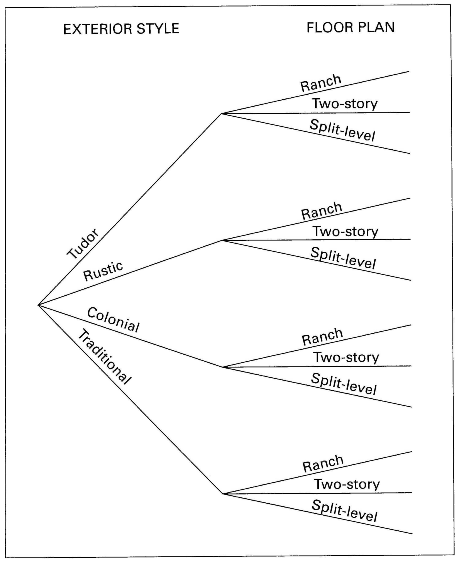

Figure 4: Tree diagram for Example 2.14.

- **Example 2.16**: How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?
- We consider the unit position by two parts, 0 or not 0.
  - If the units position is 0 ($n_1 = 1$):
    - $n_2 = 5$ choices for the thousands positions,
    - $n_3 = 4$ choices for the hundreds positions,
    - $n_4 = 3$ choices for the tens positions.
    - a total of $n_1 n_2 n_3 n_4 = 60$ choices.
  - If the units position is not 0 ($n_1 = 2$):
    - $n_2 = 4$ choices for the thousands positions,
    - $n_3 = 4$ choices for the hundreds positions,
    - $n_4 = 3$ choices for the tens positions.
    - a total of $n_1 n_2 n_3 n_4 = 96$ choices.
  - The total number of even four-digit numbers is $60 + 96 = 156$
- **Permutation**: **Definition 2.7**

  A permutation is an arrangement of all or part of a set of objects.
- An **ordered** arrangement of distinct objects. Consider the number of ways of filling r boxes with n objects.

- **Theorem 2.3**:

The number of permutation of n objects is $n!$.

- **Theorem 2.4**:

  The number of permutation (ways to arrange) of n distinct objects taken r at a time is

  $$_nP_r = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

- **Example 2.17**: In one year, three awards (research, teaching, and service) will be given for a class of 25 graduate students in a statistics department.
  - If each student can receive at most one award, how many possible selections are there?
  - Since the awards are **distinguishable**, it is a permutation problem.
  - The number of sample points is $_{25}P_3 = \frac{25!}{22!}$

- **Example 2.18**: A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if
  - there are no restrictions; $_{50}P_2 = \frac{50!}{48!} = 2450$
  - A will serve only if he is president;
    1. A is selected as the president, which yields 49 possible outcomes; or
    2. Officers are selected from the remaining 49 people which has the number of choices $_{49}P_2$

    Therefore, the total number of choices is $49 +_{49} P_2 = 2401$.
  - B and C will serve together or not at all;
    1. The number of selections when B and C serve together is 2.
    2. The number of selections when both B and C are not chosen is $_{48}P_2$

    Therefore, the total number of choices in this situation is $2 + 2256 = 2258$.
  - D and E will not serve together; $2 * 48 + 2 * 48 +_{48} P_2$

1. The number of selections when D serves as officer but not E,
2. The number of selections when E serves as officer but not D
3. The number of selections when both D and E are not chosen

Therefore, the total number of choices is 2448. This problem also has another short solution: ${}_{50}P_2 - 2$ (since D and E can only serve together in 2 ways).

- Permutations are used when we are sampling **without replacement** and **order matters**.

- **Theorem 2.5**:

  The number of permutation of n objects arranged in a circle is $(n-1)!$

- Permutations that occur by arranging objects in a circular are called **circular permutations**.

- Two circular permutations are not considered different unless corresponding objects in the two arrangements are preceded or followed by a different objects as we proceed in a clockwise direction.

- **Theorem 2.6**:

  The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, ..., n_k of a k^{th} kind is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

- **Example 2.19**: In a college football training session, the defensive coordinator needs to have 10 players standing in a row.

  - Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors, respectively.
  - How many different ways can they be arranged in a row if only their class level will be distinguished?

$$\frac{10!}{1!2!3!4!} = 12600$$

- **Theorem 2.7**:

  The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!n_2!\dots n_r!}$$

- The order of the elements within each cell is of no importance.
- The intersection of any two cells is the empty set and the union of all cells gives the original set.
- **Example 2.22**: How many different letter arrangements can be made from the letters in the word of STATISTICS?
- We have total 10 letters, while letters S and T appear 3 times each, letter I appears twice, and letters A and C appear once each.

$$\binom{10}{3,3,2,1,1} = \frac{10!}{3!3!2!1!1!} = 50400$$

- **Theorem 2.8**:

> The number of combinations (ways of choosing, **regardless of order**) of n distinct objects taken r at a time is
>
> $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- We might want to select r objects from n without regard to order.
- These selections are called **combinations**. Combinations are used when we are sampling **without replacement** and **order does NOT matter**.
- A combination is a partition with two cells,
  - the one containing r objects selected
  - the one containing the $(n-r)$ objects that are left
- The number of such combinations,

$$\binom{n}{r, n-r} \Longrightarrow \binom{n}{r}$$

- The number of permutations of n distinct objects is $n!$.
- The number of permutations of n distinct objects taken r at a time is

$$P(n,r) = \frac{n!}{(n-r)!}$$

- The number of permutations of n distinct objects arranged in a circle is

$$\frac{n!}{n} = (n-1)!$$

- The number of permutations of n things of which n_1 are of one kind, n_2 of a second kind, ..., and n_k of a k^{th} kind is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

- The number of arrangements of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\left(\begin{array}{c} n \\ n_1, n2, \dots, n_r \end{array} \right) = \frac{n!}{n_1!n_2!\dots n_r!}, (where\ n_1 + n_2 + \dots + n_r = n)$$

- The number of combinations of n distinct objects taken r at a time is

$$C(n,r) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$$

1.4 Probability of Event

- Perhaps it was man's unquenchable thirst for gambling that led to the early development of probability theory.
- What do we mean when we make the statements
  - John will probably win the tennis match.
  - I have a fifty-fifty chance of getting an even number when a die is tossed.
  - I am not likely to win at bingo tonight.
  - Most of our graduating class will likely be married within 3 years.
- In each case, we are expressing an outcome of which we are not certain, but owing to past information or from an understanding of the structure of the experiment, we have some degree of confidence in the validity of the statement.
- The likelihood of the occurrence of an event resulting from a statistical experiment is evaluated by means of a set of real numbers called **weights** or **probabilities** range from 0 to 1.
- The probability is a numerical measure of the likelihood of occurrence of an event, denoted by P.

- To every point in the sample space we assign a probability such that **the sum of all probabilities is 1**.

- In many experiments, such as tossing a coin or a die, all the sample points have the same chance of occurring and are assigned equal probabilities.

- For points outside the sample space, i.e., for simple events that cannot possibly occur, we assign a probability of zero.

- **Definition 2.8**:

  The probability of an event A is the sum of the weights of all sample points in A.

  $$0 \leq P(A) \leq 1,\ P(\varnothing) = 0,\ \textit{and}\ P(S) = 1$$

  If $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

  $$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

- In fact, P is a *probability set function* of the outcomes of the random experiment, which tells us how the probability is distributed over various subsets A of a sample space S.

- **Example 2.23**: A coin is tossed twice.
  - What is the probability that at least one head occur?
  - We assign a probability w to each sample point. Then $4w = 1$.

  $$S = \{HH, HT, TH, TT\},\ A = \{HH, HT, TH\},\ \textit{and}$$

  $$P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

- **Example 2.4**: A die is loaded in such a way that an even number is twice as likely to occur as an odd number.

- If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - We assign a probability of w to each odd number and a probability $2w$ to each even number.

- Since $P(S) = 1$, $w+2w+w+2w+w+2w = 9w = 1 \Longrightarrow w = 1/9$
- $P(A) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$

- **Theorem 2.9**:

  If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}$$

- **Example 2.27**: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

$$P(C) = \frac{C(4,2) * C(4,3)}{C(52,5)} = \frac{\frac{4!}{2!2!} * \frac{4!}{3!1!}}{\frac{52!}{5!47!}} = \frac{24}{2598960} = 0.9x10^{-5}$$

- If the outcomes of an experiment are not equally likely to occur, the probabilities must be assigned based on prior knowledge or experimental evidence.

- According to the relative frequency definition of probability, the true probabilities would be the fractions of events that occur in the long run.

- The use of intuition, personal beliefs, and other indirect information in arriving at probabilities is referred to as the subjective definition of probability.