Lecture 6
Mathematical Expectation
Lecture Information
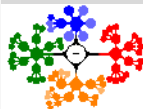
Ceng272 *Statistical Computations* at March 22, 2010
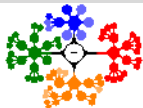
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1 Mathematical Expectation

Mean of a Random Variable

Variance and Covariance

Means and Variance of Linear Combinations of Random Variables

Chebyshev's Theorem

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.

- For a measure of central tendency of the random variable X we use the terms **expectation**, **expected value**, and **average value** for the same concept.

- Intuitively, the expected value of X is the average value that the random variable takes on.

- However, some of the values of the random variable X could be more (or less) probable than the other in the distribution unless the random variable is distributed uniformly.

- Hence, in order to consider an **average** value of X we need to take its probability into account.

Mean of a Random Variable II

- If I repeat the experiment many times, what would be the average number of an outcome of a random variable?

- **Definition 4.1**:

  Let X be a random variable with probability distribution $f(x)$. The **mean** or **expected values** of X is

  $$\left\{ \begin{array}{l} \mu = E(X) = \sum_x xf(x) \text{ if } X \text{ is discrete} \\ \mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \text{ if } X \text{ is continuous} \end{array} \right\}$$

- The expected value is used as a measure of centering or location of the distribution of a random variable X.

- By the uniform distribution assumption, i.e. all values of X are equally likely to occur in population with size N, $f(x) = \frac{1}{N}$ for all x,

$$E(X) = \sum_x xf(x) = \sum_x x(\frac{1}{N}) = (\frac{1}{N}) \sum_i x_i = \mu = \bar{x}$$
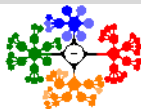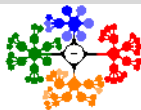
Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.

- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.

- The average number of heads per toss is then

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16}$$

x	0	1	2
$f(x)$	4/16	7/16	5/16

where $\frac{4}{16}, \frac{7}{16}, \frac{5}{16}$ are relative frequencies

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable IV

- **Example 4.1**: A lot contain 4 good components and 3 defective components.
  - A sample of 3 is taken by a quality inspector.
  - Find the expected value of the number of good components in this sample.

- Solution: X represents the number of good components

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0, 1, 2, 3$$

$$\mu = E(X) = 0 * f(0) + 1 * f(1) + 2 * f(2) + 3 * f(3) = \frac{12}{7}$$

Mean of a Random Variable V

- **Example 4.3**: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is as the following.
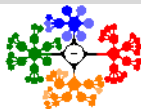
$$f(x) = \left\{ \begin{array}{ll} \frac{20000}{x^3}, & x > 100 \\ 0, & elsewhere \end{array} \right\}$$

  Find the expected life of this type of device.

- Solution:

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = -\frac{20000}{x} \Big|_{100}^{\infty} = 200$$

Mean of a Random Variable VI

- **Mean of** $g(X)$ (any real-valued function): If X is a discrete random variable with $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$ then

  $P[g(X) = 0] = P(X = 0) = f(0),$
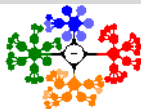 $P[g(X) = 1] = P(X = -1) + P(X = 1) = f(-1) + f(1),$
  $P[g(X) = 4] = P(X = 2) = f(2),$

- The probability distribution of $g(X)$ can be written

$g(x)$	0	1	4
$P[g(X) = 4]$	f(0)	f(-1)+f(1)	f(2)

-

$$
\begin{aligned}
E(g(X)) &= 0 * f(0) + 1 * [f(-1) + f(1)] + 4 * f(2) \\
&= (-1)^2 * f(-1) + (0)^2 * f(0) + (1)^2 * f(1) + (2)^2 * f(2) \\
&= \sum_x g(x) * f(x)
\end{aligned}
$$

Mean of a Random Variable VIII

- **Theorem 4.2:**:

  Let X and Y be random variables with joint probability function $f(x, y)$. The mean of the random variable $g(X, Y)$ is

  $$\left\{ \begin{array}{l} \mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y)f(x,y) \\ \text{if } X \text{ and } Y \text{ are discrete} \\ \mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy \\ \text{if } X \text{ and } Y \text{ are continuous} \end{array} \right\}$$

- **Example 4.7**: Find $E(Y/X)$ for the density function

  $$f(x,y) = \left\{ \begin{array}{ll} \frac{x(1+3y^2)}{4}, & 0 < x < 2, \, 0 < y < 1 \\ 0, & elsewhere \end{array} \right\}$$

- Solution:

  $$E(\frac{Y}{X}) = \int_0^1 \int_0^2 \frac{y}{x} \frac{x(1+3y^2)}{4} dxdy = \frac{5}{8}$$

Mean of a Random Variable IX

- If $g(X, Y) = X$ is

$$E(X) = \left\{ \begin{array}{l} \sum_x \sum_y x f(x, y) = \sum_x x g(x) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx dy = \int_{-\infty}^{\infty} x g(x) dx \end{array} \right\}$$

  where $g(x)$ is the marginal distribution of X

- If $g(X, Y) = Y$ is

$$E(Y) = \left\{ \begin{array}{l} \sum_x \sum_y y f(x, y) = \sum_y y h(y) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) dx dy = \int_{-\infty}^{\infty} y h(y) dy \end{array} \right\}$$

  where $h(y)$ is the marginal distribution of Y

Variance and Covariance I

- A mean does not give adequate description of the shape of a random variable (probability distribution).
- We need to characterize the variability (or dispersion) of the random variable X in the distribution.
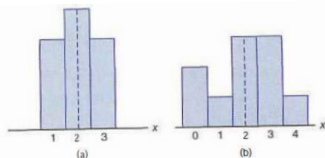
Figure: Distributions with equal means and unequal dispersions.

- **Definition 4.3**:

  Let X be a random variable with probability distribution $f(x)$ and mean μ. The **variance** of X is

  $$\begin{cases} \sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x), \text{ if } X \text{ is discrete} \\ \sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \text{ if } X \text{ is continuous} \end{cases}$$

  σ is called the **standard deviation** of X.
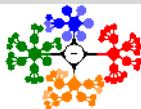
Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

x	1	2	3
$f(x)$	0.3	0.4	0.3

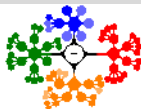| x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

- Show that the variance of the probability distribution for company B is greater than that of company A.
- Solution:

$$\mu_A = E(X) = 1*0.3 + 2*0.4 + 3*0.3 = 2.0$$

$$\sigma_A^2 = \sum_{x=1}^{3}(x-2.0)^2 f(x) = (1-2)^2*0.3+(2-2)^2*0.4+(3-2)^2*0.3 = 0.6$$

$$\mu_B = 2.0 \ \& \ \sigma_B^2 = 1.6$$

Variance and Covariance III

- **Theorem 4.2**:

  The **variance** of a random variable X is

  $$\sigma^2 = E(X^2) - \mu^2$$

- **Example 4.9**: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested.
- Calculate σ^2 using the following probability distribution.

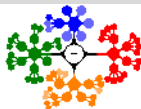| x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

- Solution:

$$\mu = E(X) = 0 * 0.51 + \ldots = 0.61$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = 0^2 * 0.51 + \ldots = 0.87$$

$$\sigma^2 = E(X^2) - \mu^2 = 0.87 - 0.61^2 = 0.4979$$

Mathematical Expectation

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Mathematical Expectation

Mean of a Random Variable

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Variance and Covariance IV

- **Theorem 4.3**:

  Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

  $$\left\{ \begin{array}{l} \sigma_{g(X)}^2 = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \sum_x [g(X) - \mu_{g(X)}]^2, \\ \text{if } X \text{ is discrete} \\ \sigma_{g(X)}^2 = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \int_{-\infty}^{\infty} [g(X) - \mu_{g(X)}]^2 f(x) dx, \\ \text{if } X \text{ is continuous} \end{array} \right\}$$

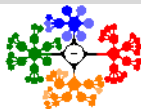- **Example 4.11**: Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution.

x	0	1	2	3
$f(x)$	1/4	1/8	1/2	1/8

- Solution:

$$\mu_{2X+3} = E(2X + 3) = \sum_{x=0}^{3} (2X + 3) f(x) = 6$$

$$\sigma_{2X+3}^2 = E\left\{[2X + 3 - \mu_{2X+3}]^2\right\} = E\left\{[2X + 3 - 6]^2\right\}$$

$$= E(4X^2 - 12X + 9) = \sum_{x=0}^{3} (4X^2 - 12X + 9) f(x) = 4$$
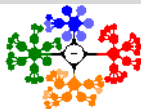
Variance and Covariance V

- **Definition 4.4**:

  Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

  $$\begin{cases} \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y), \\ \text{if } X \text{ and } Y \text{ are discrete} \\ \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y)dxdy, \text{ if } X \text{ and} Y \text{ are continuous} \end{cases}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.

- The **sign** of the covariance indicates whether the relationship between two dependent random variables is positive or negative.

- When X and Y are statistically independent, it can be shown that the covariance is zero.

- The converse, however, is not generally true. Two variables may have zero covariance and still not be statistically independent.

Variance and Covariance VI

- The covariance only describe the linear relationship between two random variables.
- If a covariance between X and Y is zero, X and Y may have a nonlinear relationship, which means that they are not necessarily independent.
- **Theorem 4.4**:

  The covariance of two random variables X and Y with means μ_X and μ_Y respectively, is given by

  $$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

Variance and Covariance VII

- **Definition 4.5**:

  Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The **correlation coefficient** of X and Y is

  $$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, \; -1 \leq \rho_{XY} \leq 1$$

- Exact linear dependency: $Y = a + bX$

  $$\rho_{XY} = 1, \; \text{if } b > 0 \;\; ; \;\; \rho_{XY} = -1, \; \text{if } b < 0$$

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.
- These properties will permit us to deal with expectations in terms of other parameters that are either known or are easily computed.
- **Theorem 4.5**:

  If a and b are constants, then

  $$E(aX + b) = aE(X) + b$$

- Corollary 4.1: $E(b) = b$
- Corollary 4.2: $E(aX) = aE(X)$

Means and Variance of Linear Combinations of Random Variables II

- **Example 4.16**: Applying Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$, the density function of X is as follows.

$$f(x) = \left\{ \begin{array}{l} \frac{x^2}{3} \ for \ -1 < x < 2 \\ 0, \ elsewhere \end{array} \right\}$$

- Solution:

$$E(4X + 3) = 4E(X) + 3 = 4 \left(\int_{-1}^{2} x \frac{x^2}{3} dx \right) + 3 = 8$$

- **Theorem 4.6**:

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

  $$E[g(X, Y) \pm h(X, Y)] = E[g(X, Y)] \pm E[h(X, Y)]$$

- Corollary 4.3: Setting $g(X, Y) = g(X)$ and $h(X, Y) = h(Y)$.

  $$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X, Y) = X$ and $h(X, Y) = Y$.
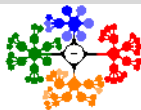
  $$E[X \pm Y] = E(X) \pm E(Y)$$

- **Theorem 4.7**:

  Let X and Y be two independent random variables. Then

  $$E(XY) = E(X)E(Y)$$

- Corollary 4.5: Let X and Y be two independent random variables, Then $\sigma_{XY} = 0$
  - $E(XY) = E(X)E(Y)$ for independent variables
  - $\sigma_{XY} = E(XY) - E(X)E(Y) = 0$

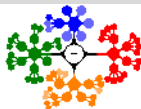# Means and Variance of Linear Combinations of Random Variables IV

- **Example 4.19**: In producing gallium-arsenide microchips, it is known that the ratio between gallium and arsenide is independent of producing a high percentage of workable wafers.
- Let X denote the ratio of gallium to arsenide and Y denote the percentage of workable wafers retrieved during a 1-hour period.
- X and Y are independent random variables with the joint density being known as

$$f(x) = \left\{ \begin{array}{l} \frac{x(1+3y^2)}{4} \text{ for } 0 < x < 2, \ 0 < y < 1 \\ 0, \text{ elsewhere} \end{array} \right\}$$

Illustrate that $E(XY) = E(X)E(Y)$.

Means and Variance of Linear Combinations of Random Variables V
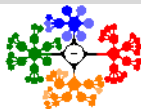
- Solution:

$$E(XY) = \int_0^1 \int_0^2 xyf(x,y)dxdy = \int_0^1 \int_0^2 xy\frac{x(1+3y^2)}{4}dxdy = \frac{5}{6}$$

$$E(X) = \int_0^1 \int_0^2 xf(x,y)dxdy = \int_0^1 \int_0^2 x\frac{x(1+3y^2)}{4}dxdy = \frac{4}{3}$$

$$E(Y) = \int_0^1 \int_0^2 yf(x,y)dxdy = \int_0^1 \int_0^2 y\frac{x(1+3y^2)}{4}dxdy = \frac{5}{8}$$

$$(E(XY) =)\frac{5}{6} = \frac{4}{3} * \frac{5}{8}(= E(X) * E(Y))$$

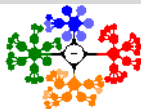# Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If *a* and *b* are constants, then

  $$\sigma^2_{aX+b} = a^2 \sigma^2_X = a^2 \sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$
  - The variance is unchanged if a constant is added to or subtracted from a random variable.
  - The addition or subtraction of a constant simply shifts the values of *X* to the right/left but does not change their variability.
- Corollary 4.7: $\sigma^2_{aX} = a^2 \sigma^2_X = a^2 \sigma^2$
  - The variance is multiplied or divided by the square of the constant.

Means and Variance of Linear Combinations of Random Variables VII

- **Theorem 4.10**:

  If X and Y are random variables with joint probability distribution $f(x, y)$, then

  $$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

- Corollary 4.8: If X and Y are independent random variables, then
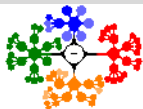
  $$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- Corollary 4.9: If X and Y are independent random variables, then

  $$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- Corollary 4.10: If $X_1, X_2, \ldots X_n$ are independent random variables, then

  $$\sigma^2_{a_1X_1+a_2X_2+\ldots a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \ldots + a_n^2\sigma^2_{X_n}$$

# Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \ (by \ Theorem \ 4.9)$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \ (by \ Theorem \ 4.10)$$
$$= 130$$

- **Example 4.21**: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product.
- Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 3$
- Find the variance of the random variable $Z = 3X - 2Y + 5$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-2Y+5}^2 = \sigma_{3X-2Y}^2 \ (by \ Theorem \ 4.9)$$
$$= 9\sigma_X^2 + 4\sigma_Y^2 \ (by \ Corollary \ 4.9)$$
$$= 30$$

Chebyshev's Theorem I
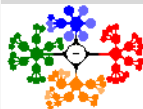
- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean
- A large variance indicates a greater variability, so the area of distribution should be spread out more.
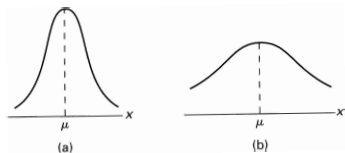

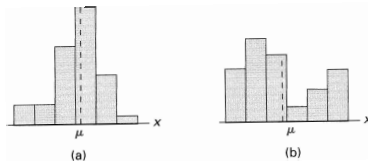**Figure:** Variability of continuous observations about the mean.

Figure: Variability of discrete observations about the: mean.

Mathematical Expectation

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Mathematical Expectation

Mean of a Random Variable

Variance and Covariance

Means and Variance of Linear Combinations of Random Variables

Chebyshev's Theorem

Chebyshev's Theorem II

- **Theorem 4.11**:

> (**Chebyshev's theorem**) The probability that any random variable X will assume a value within k standard deviation of the mean is at least $1 - 1/k^2$. That is
>
> $$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

- **Example 4.22**: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find
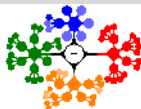
- $P(-4 < X < 20)$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(-4 < X < 20) = P(8 - 4*3 < X < 8 + 4*3) \geq 1 - \frac{1}{4^2} = \frac{15}{16}$$

- $P(|X - 8| \geq 6)$

$$P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) = 1 - P(-6 < X - 8 < 6)$$

$$= 1 - P(8 - 6 < X < 6 + 8) = 1 - P(8 - 2*3 < X < 8 + 2*3) \leq \frac{1}{2^2} = \frac{1}{4}$$

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.

- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.

- The use of Chebyshev's theorem;
  - holds for any distribution of observations
  - gives a lower bound only
  - is suitable to situations where the form of the distribution is unknown (a distribution-free result)